2.3 – Properties of Determinants; Cramer's Rule

Properties of Determinants

- For an $n \times n$ matrix A, $\det(kA) = k^n \det(A)$

- **Lemma 2.3.2** (prelude to a later result)
  If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then
  $\det(EB) = \det(E)\det(B)$.

  $\iff$

- **Theorem 2.3.3** A square matrix A is invertible if and only if $\det(A) \neq 0$.

- **Theorem 2.3.4** If A and B are square matrices of the same size, then
  $\det(AB) = \det(A)\det(B)$. But $\det(A+B) \neq \det(A) + \det(B)$

- **Theorem 2.3.5** If A is invertible, then $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$.

1st bullet - one factor of k for each row.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$2R_1$: $\begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 8 - 12 = -4 = 2(-2) = \underline{2\det(A)}$

But $\underline{3A} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \Rightarrow \det(3A) = \begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix}$

$= 36 - 54 = -18 = 3^2(-2)$

Please note the difference in notation:

$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Lemma 2.3.2 – Suppose E results from multiplying the i^{th} row of I by k.

Then $EB = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} B = \begin{bmatrix} \vec{e}_1 B \\ \vec{e}_2 B \\ \vdots \\ k\vec{e}_i B \\ \vdots \\ \vec{e}_n \end{bmatrix}$,

So the i^{th} row of EB is the i^{th} row of B, multiplied by k. By Thm 2.2.3(a), $\det(E) = k$ so $\det(EB) = k \det(B)$
$= \det(E)\det(B)$

Thm 2.3.3 – A invertible $\Rightarrow$ A is a product of elementary matrices $\Rightarrow \det(A) \neq 0$ (using Thm 2.3.2).

$\det(A) \neq 0 \Rightarrow$ rref of A does not have a row of zeros (contrapositive of Thm 2.2.1)
$\Rightarrow \text{rref}(A) = I \Rightarrow A$ is invertible.

Thm 2.3.4 - If A and/or B is not invertible, then AB is not invertible $\Rightarrow \det(A)=0$ or $\det(B)=0$ and $\det(AB)=0 \Rightarrow 0=0$
$\Rightarrow \det(AB)=\det(A)\det(B)$

(outline of proof)

If A & B are invertible, we use products of elementary matrices and lemma 2.3.2.

Thm 2.3.5 - A invertible $\Rightarrow A^{-1}$ exists and $\det(A)\neq 0$,

$AA^{-1}=I$

$\Rightarrow \det(AA^{-1}) = \det(I) = 1$

$\Rightarrow \det(A)\det(A^{-1}) = 1$ (Thm 2.3.4)

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

Theorem 2.3.8 Equivalent Statements (extends Theorem 1.6.4)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

#16 Find the values of k for which the matrix A is invertible.

$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

$\det(A) = 0 \Rightarrow k^2 - 4 = 0 \Rightarrow k = \pm 2$

A is invertible for $k \neq \pm 2$

#35 In each part, find the determinant given that A is a 3×3 matrix for which $\det(A) = 7$.

a. $\det(3A)$

b. $\det\left(A^{-1}\right)$

c. $\det\left(2A^{-1}\right)$

d. $\det\left((2A)^{-1}\right)$

a) $\det(3A) = 3^3 \det(A) = 189$

b) $\det(A^{-1}) = \frac{1}{7}$

c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{7}$

d) $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{56}$

Definition: If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$ is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $\mathrm{adj}(A)$.

Theorem 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A)$.

#20 Decide whether the matrix is invertible, and if so, use the adjoint to find its inverse.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\det(A) = 2(-12) + 0(-4) + 3(6) = -6 \neq 0 \Rightarrow A \text{ is invertible}$$

$$M_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12 \qquad M_{12} = \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = 4 \qquad M_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$C_{11} = -12$ $\qquad C_{12} = -4$ $\qquad C_{13} = 6$

$C_{21} = 0$ $\qquad C_{22} = -2$ $\qquad C_{23} = 0$

$C_{31} = -9$ $\qquad C_{32} = -4$ $\qquad C_{33} = 6$

Matrix of cofactors: $[C_{ij}] = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

Transpose of $[C_{ij}]$ is the adjoint of A.

$$[C_{ij}]^T = \operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{6} \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$$

Theorem 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is
$x_1 = \dfrac{\det(A_1)}{\det(A)}$, $x_2 = \dfrac{\det(A_2)}{\det(A)}$, $\ldots$, $x_n = \dfrac{\det(A_n)}{\det(A)}$ where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

#26 Solve by Cramer's rule.

$$x - 4y + z == 6$$
$$4x - y + 2z = -1$$
$$2x + 2y - 3z = -20$$

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$-2 + 4 + 48 = 50$

$$\det(A) = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix} \qquad -5 - 50 = -55$$

$3 - 16 + 8 = -5$

$$\det(A_1) = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 144$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 61$$

$$\det(A_3) = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = -230$$

$$x = \frac{\det(A_1)}{\det(A)} = -\frac{144}{55}$$

$$y = \frac{\det(A_2)}{\det(A)} = -\frac{61}{55}$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{230}{55} = \frac{46}{11}$$

$$(x, y, z) = \left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)$$

<u>pf of Cramer's Rule</u>: The existence of a unique solution is guaranteed by Thm 2.3.8 (Equivalent Statements).

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b} = \frac{1}{\det(A)} \operatorname{adj}(A)\, \vec{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} \vec{b} \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 C_{21} + \cdots b_n C_{n1} \\ b_2 C_{12} + b_2 C_{22} + \cdots + b_n C_{n1} \\ \vdots \qquad \vdots \qquad \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{bmatrix}$$

Then $x_i = \dfrac{b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}}{\det(A)}$

The numerator is the expansion along the i^{th} column of A, replaced by $\vec{b}$.