3.2 – Norm, Dot Product, and Distance in R^n

Definition: If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** of $\mathbf{v}$ (also called its **length** or **magnitude**) is denoted by $\|\mathbf{v}\|$ [by this author], and is defined by the formula $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.

Theorem 3.2.1 Properties of the norm of a vector

If $\mathbf{v}$ is a vector in R^n and k is any scalar, then:

a) $\|\mathbf{v}\| \geq 0$

b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

c) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$

Let $\vec{v} = (v_1, v_2, \ldots, v_n)$.

$k\vec{v} = (kv_1, kv_2, \ldots, kv_n)$ def of scalar mult.

$\|k\vec{v}\| = \sqrt{k^2v_1^2 + k^2v_2^2 + \cdots + k^2v_n^2}$ def of norm

$= |k|\,\|\vec{v}\|$ def of abs value

Definition: The norm of a **unit vector** is 1. We can obtain a unit vector from a nonzero vector $\mathbf{v}$ by multiplying by the reciprocal of its length. This process is called **normalizing** the vector.

$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{\vec{v}}{\|\vec{v}\|} = \vec{u}$ is a unit vector

#1 Find the norm of $\mathbf{v}$ and a unit vector that is oppositely directed to $\mathbf{v}$.

a. $\mathbf{v} = (2,\ 2,\ 2)$

b. $\mathbf{v} = (1,\ 0,\ 2,\ 1,\ 3)$

a) $\vec{v} = (2,2,2) = 2(1,1,1) \Rightarrow \|\vec{v}\| = 2\sqrt{1^2+1^2+1^2} = 2\sqrt{3}$

$-\frac{1}{2\sqrt{3}}\vec{v} = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ has length 1 and is oppositely directed to $\vec{v}$.

b) $\|\vec{v}\| = \sqrt{1+4+1+9} = \sqrt{15}$

$-\frac{1}{\sqrt{15}}(1,0,2,1,3) = \left(-\frac{1}{\sqrt{15}}, 0, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$

Definition: The **standard unit vectors in $\mathbf{R}^n$** are the standard basis vectors for R^n, $\mathbf{e}_1 = (1,\ 0,\ 0,\ \ldots,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0,\ \ldots,\ 0)$, $\ldots$, $\mathbf{e}_n = (0,\ 0,\ 0,\ \ldots,\ 1)$.

Recall: In R^2, $\vec{e}_1 = (1,0) = \hat{i}$, $\vec{e}_2 = (0,1) = \hat{j}$

R^3: $\vec{e}_1 = (1,0,0) = \hat{i}$, $\vec{e}_2 = (0,1,0) = \hat{j}$, $\vec{e}_3 = (0,0,1) = \hat{k}$

Every vector in R^n can be expressed as a linear combination of the standard basis vectors:

$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

Definition: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the **distance** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be
$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$

#4 Evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

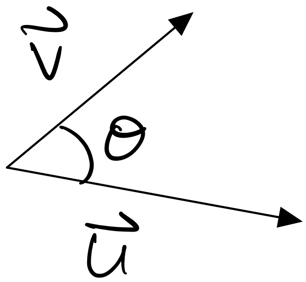
a. $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$
b. $\|\mathbf{u} - \mathbf{v}\|$ ← This is the distance between $\vec{u}$ & $\vec{v}$
c. $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$
d. $\|\mathbf{u}\| - \|\mathbf{v}\|$

b) $\vec{u} - \vec{v} = \sqrt{(2-1)^2 + (-2+3)^2 + (3-4)^2} = \sqrt{1+1+1} = \sqrt{3}$

c) $\|3\vec{v}\| - 3\|\vec{v}\| = 3\|\vec{v}\| - 3\|\vec{v}\| = 0$

d) $\|\vec{u}\| - \|\vec{v}\| = \sqrt{4+4+9} - \sqrt{1+9+16}$
$= \sqrt{17} - \sqrt{26}$

Definition: Let **u** and **v** be nonzero vectors in R^n positioned so that their initial points coincide. The **angle between u and v** is the angle θ determined by **u** and **v** such that $0 \le \theta \le \pi$.



$3(x)$

$3x$

$3 \times x$

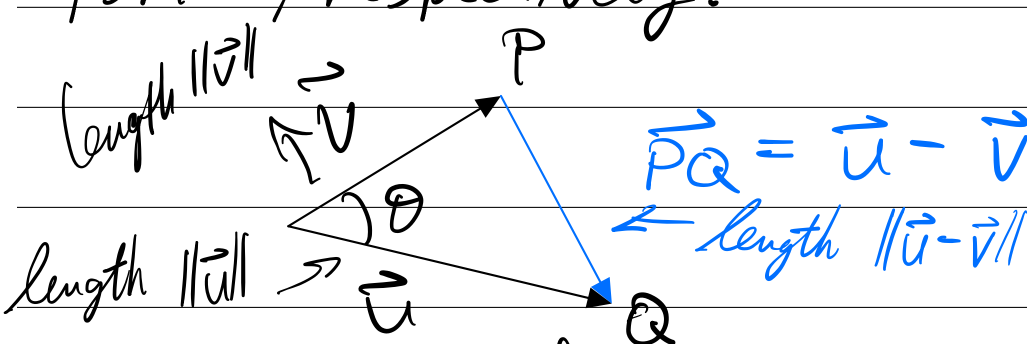
$3 \cdot x$

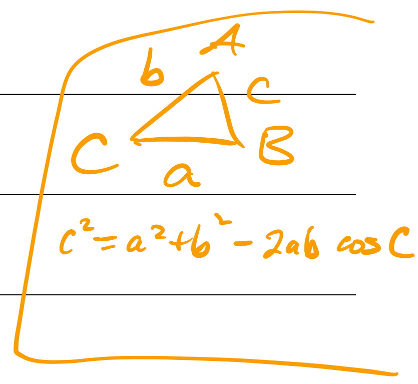
Definition: If **u** and **v** are nonzero vectors in R^2 or R^3, and if θ is the angle between **u** and **v**, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

↑ The dot is required

Definition: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

To connect the two definitions, let $\vec{u}$ & $\vec{v}$ share an initial point in R^3 and let Q, P be their terminal points, respectively.





By the Law of Cosines

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\Rightarrow (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = \cancel{u_1^2} + u_2^2 + u_3^2 + \cancel{v_1^2} + v_2^2 + v_3^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2}$$

$$\cancel{-2}(u_1v_1+u_2v_2+u_3v_3) = \cancel{-2}\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\|\vec{u}\|\|\vec{v}\|\cos\theta = \underbrace{\underbrace{u_1v_1+u_2v_2}_{R^2}+u_3v_3}_{R^3} = \vec{u}\cdot\vec{v}$$

#10 Find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{u}$, and $\mathbf{v}\cdot\mathbf{v}$.

a. $\mathbf{u} = (1,\ 1,\ -2,\ 3)$, $\mathbf{v} = (-1,\ 0,\ 5,\ 1)$

b. $\mathbf{u} = (2,\ -1,\ 1,\ 0,\ -2)$, $\mathbf{v} = (1,\ 2,\ 2,\ 2,\ 1)$

b) $\vec{u}\cdot\vec{v} = 2(1) - 1(2) + 1(2) + 0(2) - 2(1) = 0$

$\vec{u}\cdot\vec{u} = 1^2 + 1^2 + (-2)^2 + 3^2 \quad (= \|\vec{u}\|^2)$

$= 15$

#11 Find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°.

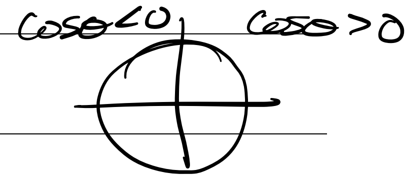
a. $\mathbf{u} = (3,\ 3,\ 3)$, $\mathbf{v} = (1,\ 0,\ 4)$

b. $\mathbf{u} = (0,\ -2,\ -1,\ 1)$, $\mathbf{v} = (-3,\ 2,\ 4,\ 4)$

a) $\|\vec{u}-\vec{v}\| = \sqrt{4+9+1} = \sqrt{14}$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$\cos\theta = \dfrac{15}{3\sqrt{3}\sqrt{17}} \Rightarrow \cos\theta = \dfrac{5}{\sqrt{51}}$



Theorem 3.2.2 Properties of the dot product

If **u**, **v**, and **w** are vectors in R^n and if k is a scalar, then:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry property)

b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive property)

c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ (homogeneity property)

d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity property)

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots, v_n)$,

$\vec{w} = (w_1, w_2, \ldots, w_n)$

def of vector addition

$\vec{u} \cdot (\vec{v} + \vec{w}) = (u_1, u_2, \ldots, u_n) \cdot (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n)$

def of dot product

$= u_1 v_1 + u_1 w_1 + \cdots + u_n v_n + u_n w_n$

dist. prop. R #s.

$= u_1 v_1 + \cdots + u_n v_n + u_1 w_1 + \cdots + u_n w_n$

Comm prop R #s

$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ def of dot product

Theorem 3.2.3 More properties of the dot product

If **u**, **v**, and **w** are vectors in R^n and if k is a scalar, then:

a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Theorem 3.2.4 Cauchy-Schwarz Inequality — a necessary result for the cosine def. of the dot product in R^n.

If $\mathbf{u} = (u_1, u_2, \ldots u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$ or in terms of components

$|u_1v_1 + u_2v_2 + \ldots + u_nv_n| \leq \left(u_1^2 + u_2^2 + \ldots u_n^2\right)^{\frac{1}{2}} \left(v_1^2 + v_2^2 + \ldots v_n^2\right)^{\frac{1}{2}}$.

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\| \Rightarrow \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$$

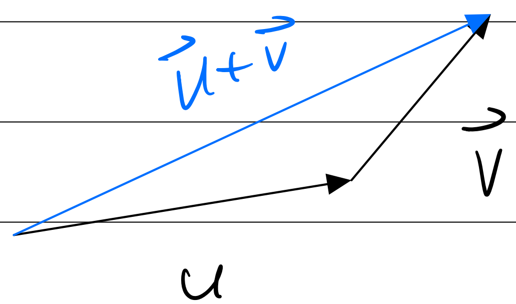
$$\Rightarrow -1 \leq \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1 \Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

Theorem 3.2.5 Triangle Inequalities

If **u**, **v**, and **w** are vectors in R^n, then:

a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b) $d\left(\mathbf{u},\ \mathbf{v}\right) \leq d\left(\mathbf{u},\ \mathbf{w}\right) + d(\mathbf{w},\ \mathbf{v})$ (triangle inequality for distances)



Pf: (a) $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$$\leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2$$

(def of norm and property of absolute value)

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2$$

(Cauchy-Schwarz inequality)

$$= \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

Taking square roots completes the proof.

b) follows from (a) and def of distance.

Note: An expression like $\vec{u} \cdot \vec{v} \cdot \vec{w}$ does not have meaning. But we can consider matrix mult. using the dot product. If A is $n \times n$, $\vec{u}, \vec{v}$ $n \times 1$,

$$A\vec{u} \cdot \vec{v} = \vec{v}^T (A\vec{u}) = (v^T A)\vec{u}$$
$$= (A^T \vec{v})^T \vec{u} = \vec{u} \cdot A^T \vec{v}$$

So $A\vec{u} \cdot \vec{v} = \vec{u} \cdot A^T \vec{v}$

Consider A $m \times n$, B $n \times r$

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix} \quad B = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_r \end{bmatrix}$$

$$AB = \begin{bmatrix} \vec{r}_1 \cdot \vec{c}_1 & \vec{r}_1 \cdot \vec{c}_2 & & \\ & & \ddots & \\ & & & \vec{r}_m \cdot \vec{c}_r \end{bmatrix}$$

False

$(2,3,4)$, $(3,-2,5)$

All vectors for which $u_1 = v_2$, $v_2 = -u_1$ satisfy $\vec{u} \cdot \vec{v} = 0$

pf: $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2) = (u_2, -u_1)$

$\vec{u} \cdot \vec{v} = u_1 u_2 - u_2 u_1 = 0$

Limits the scope of the statement we're trying to prove.