

4.3 - Homogeneous Linear Equations with Constant Coefficients

Consider the differential equation $y'' - 5y' + 6y = 0$.

Start with $y = e^{mx}$. $y' = m e^{mx}$

$$y'' = m^2 e^{mx}$$

$$m^2 e^{mx} - 5m e^{mx} + 6e^{mx} = 0$$

$$(m^2 - 5m + 6)e^{mx} = 0$$

$$\Rightarrow m^2 - 5m + 6 = 0 \quad (\text{auxiliary equation})$$

$$(m - 2)(m - 3) = 0$$

$$m = 2, 3$$

$$y_1 = e^{2x}, \quad y_2 = e^{3x}$$

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Definition: The **auxiliary equation** of the second-order linear differential equation $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$.

The auxiliary equation has solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

this leads to 3 cases:

Case 1: $m_1 \neq m_2$, both are real

then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2: $m_1 = m_2$, real (repeated roots)

If $b^2 - 4ac = 0$, $m_1 = -\frac{b}{2a}$

$$y_1 = e^{-\frac{b}{2a}x}$$

Reduction of order: $y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$

$$ay'' + by' + cy = 0 \Rightarrow y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}} dx = x y_1$$

Then $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

Case 3: $b^2 - 4ac < 0 \Rightarrow m_1 = \alpha + \beta i, m_2 = \alpha - \beta i$

$$y = c_1 e^{(\alpha + \beta i)x} + c_2 e^{(\alpha - \beta i)x}$$

$$y = e^{\alpha x} (c_1 e^{\beta i x} + c_2 e^{-\beta i x}) \quad \text{for all } c_1, c_2$$

Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$

[side note: If $\theta = \pi \Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$]

$$e^{\beta i x} = \cos \beta x + i \sin \beta x$$

$$e^{-\beta i x} = \cos \beta x - i \sin \beta x$$

$$e^{\beta i x} + e^{-\beta i x} = 2 \cos \beta x \Rightarrow \cos \beta x = \frac{e^{\beta i x} + e^{-\beta i x}}{2}$$

$$\text{Likewise, } \sin \beta x = \frac{e^{\beta i x} - e^{-\beta i x}}{2i}$$

Strategic choices for c_1, c_2 above give the following general solution:

If $m = \alpha \pm \beta i$, then

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Example: Find the general solution of the given second-order differential equation.

$$y'' - 10y' + 25y = 0$$

$$m^2 - 10m + 25 = 0$$

$$(m - 5)^2 = 0 \Rightarrow m = 5 \text{ (repeated)}$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

Example: $2y'' - 3y' + 4y = 0$

$$2m^2 - 3m + 4 = 0 \Rightarrow m = \frac{3 \pm \sqrt{9 - 32}}{4}$$

$$m = \frac{3}{4} \pm \frac{\sqrt{23}}{4} i$$

$$y = e^{\frac{3}{4}x} \left(c_1 \cos \frac{\sqrt{23}}{4}x + c_2 \sin \frac{\sqrt{23}}{4}x \right)$$

Example: Find the general solution of the given higher-order differential equation.

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$$

we're looking for $x(t)$

$$m^3 - m^2 - 4 = 0$$

Wolfram alpha. com $m = 2, -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$$x(t) = c_1 e^{2t} + e^{-\frac{1}{2}t} \left(c_2 \cos \frac{\sqrt{7}}{2} t + c_3 \sin \frac{\sqrt{7}}{2} t \right)$$

Special 2nd-order differential equations:

$$y'' + k^2 y = 0 \text{ and } y'' - k^2 y = 0, k \in \mathbb{R}$$

$$\rightarrow m^2 + k^2 = 0$$

$$m^2 - k^2 = 0 \Rightarrow m = \pm k$$

$$m = \pm k i$$

$$y = c_1 \cos kx + c_2 \sin kx$$

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

$$\text{If } c_1 = c_2 = \frac{1}{2},$$

$$y_1 = \cosh kx$$

$$\text{If } c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$$

$$y_2 = \sinh kx$$

Alternatively

$$y = c_1 \cosh kx + c_2 \sinh kx$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Example: Find a homogeneous linear differential equation with constant coefficients whose general solution is given.

$$y = c_1 e^{-4x} + c_2 e^{-3x}$$

$$m = -4, -3 \Rightarrow (m+4)(m+3) = 0$$

$$\Rightarrow m^2 + 7m + 12 = 0$$

$$y'' + 7y' + 12y = 0$$

Example: $y = c_1 e^{10x} + c_2 x e^{10x}$

$$m = 10, \text{ repeated} \Rightarrow (m - 10)^2 = 0$$

$$m^2 - 20m + 100 = 0$$

$$y'' - 20y' + 100y = 0$$

Example: $y = c_1 + c_2 e^{2x} \cos 5x + c_3 e^{2x} \sin 5x$

$$m = 0, 2 \pm 5i$$

$$m(m - (2 + 5i))(m - (2 - 5i)) = 0$$

$$m(\underline{m - 2 - 5i})(\underline{m - 2 + 5i}) = 0$$

$$(\underline{A} - \underline{B})(\underline{A} + \underline{B}) = A^2 - B^2$$

$$m((\underline{m - 2})^2 - (\underline{5i})^2) = 0$$

$$m(m^2 - 4m + 4 + 25) = 0$$

$$m^3 - 4m^2 + 29m = 0$$

$$y''' - 4y'' + 29y' = 0$$