4.4 - Undetermined Coefficients–Superposition Approach

The techniques in this lesson apply to nonhomogeneous DEs of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$ whose input functions (that is, $g(x)$) are constants, polynomials, exponentials, sines and cosines, and sums and products of these.

From 4.1: Consider $y'' - 6y' + 5y = \overbrace{\underbrace{-9e^{2x}}_{g_1(x)} + \underbrace{5x^2 + 3x - 16}_{g_2(x)}}^{g(x)}$

The complete (general) solution is

$$y = \underbrace{c_1 e^x + c_2 e^{5x}}_{y_c \text{ - Complementary function}} + \underbrace{\overset{y_{p_1}}{3e^{2x}} + \overset{y_{p_2}}{x^2 + 3x}}_{\text{particular solution}}$$

y_c is a solution to the associated homogeneous DE $y'' - 6y' + 5y = 0$.

$g(x)$ comprises an exponential function and a polynomial function.

Consider: $y'' - 6y' + 5y = -9e^{2x}$

Since $g_1(x)$ is exponential w/ exponent $2x$,

Guess: $y_{p_1} = Ae^{2x}$

Then $y_{p_1}' = 2Ae^{2x}$, $y_{p_1}'' = 4Ae^{2x}$

so $4Ae^{2x} - 12Ae^{2x} + 5Ae^{2x} = -9e^{2x}$

$\Rightarrow 4A - 12A + 5A = -9 \Rightarrow A = 3$

$y_{p_1} = 3e^{2x}$

$g_2(x)$ is a 2^{nd}-degree polynomial

$y_{p_2} = Ax^2 + Bx + C$

This goes into $y'' - 6y' + 5y = 5x^2 + 3x - 16$

$y_{p_2}' = 2Ax + B$, $y_{p_2}'' = 2A$

$$2A - 6(2Ax + B) + 5(Ax^2 + Bx + C) = 5x^2 + 3x - 16$$

$$(5A)x^2 + (-12A + 5B)x + (2A - 6B + 5C) = 5x^2 + 3x - 16$$

x^2 term: $5A = 5 \Rightarrow A = 1$

x terms: $-12A + 5B = 3 \Rightarrow B = 3$

Constants: $2A - 6B + 5C = -16 \Rightarrow C = 0$

$$y_{p_2} = x^2 + 3x$$

$$y_p = y_{p_1} + y_{p_2} = 3e^{2x} + x^2 + 3x$$

General solution to nonhomogeneous DE

is $y = \underbrace{c_1 e^x + c_2 e^{5x}}_{y_c} + \underbrace{3e^{2x} + x^2 + 3x}_{y_p}$

We could have: $y_p = Ae^{2x} + Bx^2 + Cx + D$

Example: Solve the given differential equation by undetermined coefficients.

$y'' - 8y' + 20y = 100x^2 - 26xe^x$

Step 1: $m^2 - 8m + 20 = 0$

$m^2 - 8m + 16 = -20 + 16$

$(m-4)^2 = -4$

$m = 4 \pm 2i$

$y_c = e^{4x}(c_1 \cos 2x + c_2 \sin 2x)$

$y_{p_1} = Ax^2 + Bx + C$ $\qquad$ $y_{p_2} = (Ax+B)e^x$

$y_{p_1}' = 2A\cancel{x}+B$

$y_{p_1}'' = 2A$

$2A - 16A\cancel{x} - 8B + 20Ax^2 + 20Bx + 20C = 100x^2$

$\ldots \quad A = 5, \; B = 4, \; C = \frac{11}{10}$

$y_{p_2}' = (A + Ax + B)e^x$

$y_{p_2}'' = (2A + Ax + B)e^x$

$2A + Ax + B - 8A - 8Ax - 8B + 20Ax + 20B = -26x$

$\ldots \quad A = -2, \; B = -\frac{12}{13}$

$y = e^{4x}(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$

Example: $y'' - 16y = 2e^{4x}$

Setting ourselves up for potential failure:

$$y_p = Ae^{4x}$$

$$y_p' = 4Ae^{4x} \Rightarrow y_p'' = 16Ae^{4x}$$

$$16Ae^{4x} - 16Ae^{4x} = 2e^{4x} \rightarrow 0 = \cancel{2e^{4x}}$$

We must always first find y_c.

$$m^2 - 16 = 0 \Rightarrow m = \pm 4$$

$$y_c = c_1 e^{4x} + c_2 e^{-4x}$$

this y_p is redundant with y_c.

To find a form of y_p that is independent with y_c, we insert factor(s) of x.

Here $y_p = Axe^{4x}$ this factor of x avoids redundancy with y_c

$$y'' - 16y = 2e^{4x}$$

$$y_p' = (A + 4Ax)e^{4x}$$

$$y_p'' = (4A + 4A + 16Ax)e^{4x} = (8A + 16Ax)e^{4x}$$

$$8A + 16Ax - 16Ax = 2 \Rightarrow A = \frac{1}{4}$$

Recall:

$$y = y_c + y_p$$

$$y = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}$$

Shifting gears a bit

In general, if $g(x) = x^2 e^{4x}$,

$$y_p = (Ax^2 + Bx + c)e^{4x}$$

Suppose y_c includes xe^{4x} (redundant)

Example: $4y'' - 4y' - 3y = \cos 2x$

$$4m^2 - 4m - 3 = 0 \Rightarrow (2m+1)(2m-3) = 0$$

$$m = -\frac{1}{2}, \frac{3}{2} \Rightarrow y_c = c_1 e^{-x/2} + c_2 e^{3/2 x}$$

$$y_p = A\cos 2x + B\sin 2x$$

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

$$-16A\cos 2x - 16B\sin 2x + 8A\sin 2x - 8B\cos 2x$$

$$-3A\cos 2x - 3B\sin 2x = \cos 2x$$

Cosine terms: $-16A - 8B - 3A = 1$

Sine terms: $-16B + 8A - 3B = 0$

$$-19A - 8B = 1 \quad\rightarrow\quad 361A + 8(19)B = -19$$

$$8A - 19B = 0 \qquad\qquad 64A - 8(19)B = 0$$

$$425A = -1 \Rightarrow A = -\frac{19}{425}$$

$$-8(19)A - 64B = 8$$

$$8(19)A - 361B = 0$$

$$-425B = 8 \Rightarrow B = -\frac{8}{425}$$

$$y = c_1 e^{-x/2} + c_2 e^{3x/2} - \frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x$$

y_p forms, assuming no redundancy

g(x)	y_p
$3x^2 + 2$	$Ax^2 + Bx + C$
$7e^{4x}$	Ae^{4x}
$8x^2e^{2x}$	$(Ax^2 + Bx + C)e^{2x}$
$3\sin 2x$	$A\sin 2x + B\cos 2x$
$x\cos 3x$	$(Ax+B)\cos 3x + (Cx+D)\sin 3x$