4.6 - Variation of Parameters

Consider the homogeneous second-order differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. The general solution to this DE is $y = c_1y_1 + c_2y_2$. When the input function of an associated nonhomogeneous DE is not restricted to the type we saw in 4.4, we consider particular solutions with coefficients that are variable functions. That is, we will find $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.

Starting with $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$\rightarrow$ std form $y'' + P(x)y' + Q(x)y = f(x)$

Complementary function (4.3)

$y_c = \underbrace{c_1\ y_1 + c_2\ y_2}$, constants

Now consider $y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$

$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$

becomes

$u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2'' + u_2y_2''$

$+ Pu_1'y_1 + Pu_1y_1' + Pu_2'y_2 + Pu_2y_2'$

$+ Qu_1y_1 + Qu_2y_2 = f(x)$

since y_1, y_2 are solutions to homog. DE,

$u_1(y_1'' + Py_1' + Qy_1) = 0$ Likewise for y_2.

$u_1''y_1 + 2u_1'y_1' + Pu_1'y_1$ $\quad u_2''y_2 + u_2'y_2' + u_2'y_2'$

$u_1''y_1 + u_1'y_1' + u_1'y_1'$

$\frac{d}{dx}(u_1'y_1 + u_2'y_2)$

The above equation becomes

$$\frac{d}{dx}(u_1'y_1 + u_2'y_2) + u_1'y_1' + u_2'y_2' + P(u_1'y_1 + u_2'y_2) = f(x)$$

If we require $u_1'y_1 + u_2'y_2 = 0$,

$$u_1'y_1' + u_2'y_2' = f(x)$$

Recall Cramer's Rule:

$$ax + by = e \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
$$cx + dy = f$$

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \quad \text{if } D \neq 0$$

Here we form $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Requires standard form

$$u_1' = \frac{W_1}{W} \Rightarrow u_1 = \int \frac{W_1}{W} dx$$

$$u_2' = \frac{W_2}{W} \Rightarrow u_2 = \int \frac{W_2}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= -f(x)y_2 \qquad\qquad = f(x)y_1$$

$$u_1' = \frac{W_1}{W} \Longrightarrow u_1 = \int \frac{W_1}{W} dx$$
$$u_2' = \frac{W_2}{W} \Longrightarrow u_2 = \int \frac{W_2}{W} dx$$

$y_p = u_1y_1 + u_2y_2$ and $y = y_c + y_p$

Example: Solve each differential equation by variation of parameters.
$y'' + y = \sec\theta\ \tan\theta$

$m^2+1=0 \Rightarrow m = \pm i \Rightarrow y_c = C_1\cos\theta + C_2\sin\theta$

$y_1 = \cos\theta \qquad y_2 = \sin\theta$

$$W = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta\tan\theta & \cos\theta \end{vmatrix} = -\tan^2\theta$$

$$W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta\tan\theta \end{vmatrix} = \tan\theta$$

$$u_1 = \int -\tan^2\theta \, d\theta = \int (1 - \sec^2\theta)\, d\theta$$
$$= \theta - \tan\theta$$

$$u_2 = \int \tan\theta \, d\theta = -\int \frac{-\sin\theta}{\cos\theta}\, d\theta = - \ln|\cos\theta|$$
$$= \ln|\sec\theta|$$

$$y_p = u_1 y_1 + u_2 y_2 = (\theta - \tan\theta)\cos\theta + \sin\theta \ln|\sec\theta|$$
$$= \theta\cos\theta - \sin\theta + \sin\theta\ln|\sec\theta|$$

Matches y_2

$$y = c_1\cos\theta + c_2\sin\theta + \theta\cos\theta + \sin\theta\ln|\sec\theta|$$

Std form before identifying $f(x)$

Example: Solve the initial-value problem by variation of parameters.

$2y'' + y' - y = x + 1 \qquad y(0) = 1,\ y'(0) = 0$

$$(2m^2 + m - 1) = 0 \Rightarrow (2m-1)(m+1) = 0$$
$$m = \tfrac{1}{2}, -1 \Rightarrow y = c_1 e^{x/2} + c_2 e^{-x}$$
$$y_1 = e^{x/2}, \quad y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^{x/2} & e^{-x} \\ \frac{1}{2}e^{x/2} & -e^{-x} \end{vmatrix} = -e^{-x/2} - \frac{1}{2}e^{-x/2} = -\frac{3}{2e^{x/2}}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{2}x+\frac{1}{2} & -e^{-x} \end{vmatrix} = -\frac{1}{2e^{x}}(x+1)$$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2}e^{x/2} & \frac{1}{2}x+\frac{1}{2} \end{vmatrix} = \frac{e^{x/2}}{2}(x+1)$$

$$u_1 = \int \frac{W_1}{W}\,dx = \frac{1}{3}\int (x+1)e^{-x/2}\,dx$$

$$u_2 = \int \frac{W_2}{W}\,dx = -\frac{1}{3}\int (x+1)e^{x}\,dx$$

$$\ldots \quad u_1 = -\frac{2}{3}x e^{-x/2} - 2e^{-x/2}$$

$$u_2 = -\frac{1}{3}x e^{x}$$

$$y = y_c + y_p \; ; \; y_p = u_1 y_1 + u_2 y_2$$

$$y = c_1 e^{x/2} + c_2 e^{-x} \overset{-x-2}{\cancel{-\frac{2}{3}x - 2 - \frac{1}{3}x}}$$

$y(0) = 1$, $y'(0) = 0$

$$1 = c_1 + c_2 - 2 \qquad 0 = \frac{1}{2}c_1 - c_2 - 1$$

$$\ldots \quad \boxed{y = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2}$$

Example: Solve the differential equation by variation of parameters.

$$y'' - 4y = \frac{e^{2x}}{x}$$

$m^2 - 4 = 0 \Rightarrow m = \pm 2$

$y_1 = e^{2x}, \quad y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ e^{2x}/x & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & e^{2x}/x \end{vmatrix} = \frac{e^{4x}}{x}$$

$$u_1 = \int \frac{W_1}{W}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln|x|$$

$$u_2 = \int \frac{W_2}{W}\,dx = -\frac{1}{4}\int \frac{e^{4x}}{x}\,dx$$

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}e^{2x}\ln|x| - \frac{1}{4}e^{-2x}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$$

Example: Solve the given third-order differential equation by variation of parameters.

$y''' + 4y' = \sec 2x$

$$m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0$$

$$m = 0, \pm 2i$$

$$y_c = C_1 + C_2 \cos 2x + C_3 \sin 2x$$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} = 8$$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix} = 2\sec 2x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = -2$$

$$W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = -2\tan 2x$$

$$u_1 = \int \frac{1}{4} \sec 2x \, dx = \frac{1}{8} \ln|\sec 2x + \tan 2x|$$

$$u_2 = \int -\frac{1}{4} \, dx = -\frac{1}{4}x$$

$$u_3 = \int -\frac{1}{4} \tan 2x \, dx = \frac{1}{8} \ln|\cos 2x|$$

$$y = c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{8} \ln|\sec 2x + \tan 2x|$$
$$- \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x \ln|\cos 2x|$$