Differential operator: For $y=y(x)$

$Dy = \frac{dy}{dx}$ and $D^2 y = \frac{d^2y}{dx^2}$.

$y' + 5y$ can be written as $Dy + 5y = (D+5)y$

We can operate on this using D:

$$D(D+5)y = D(Dy+5y) = D^2y + 5Dy$$
$$= (D^2+5D)y$$

Further, we can operate on $(D+5)y$ using $D+2$:

$$D(D+5)y + 2(D+5)y$$
$$(D^2+5D)y + (2D+10)y$$
$$(D^2+7D+10)y = (D+2)(D+5)y$$

$$y' + 5y = e^{3x} \Rightarrow Dy + 5y = e^{3x}$$

$$(D+2)(D+5)y = (D+2)e^{3x} = \frac{d}{dx}(e^{3x}) + 2e^{3x}$$
$$= 3e^{3x} + 2e^{3x} = 5e^{3x}$$

4.9 - Solving Systems of Linear DEs by Elimination

The DE $y' + 2y + 3x = -1$ can be written in terms of the differential operator D, where D represents a derivative. In this form, the DE becomes $Dy + 2y + 3x = -1$. This can also be written $(D+2)y + 3x = -1$. Our work in this lesson makes use of this notation. For this lesson, x and y are functions of t, so $Dx = \dfrac{dx}{dt}$, $Dy = \dfrac{dy}{dt}$, and $D^2x = \dfrac{d^2x}{dt^2}$, etc.

Solve the given system of differential equations by elimination.

Example:

$$\begin{aligned} (1)\quad (D+1)x &+ (D-1)y = 2 \\ (2)\quad 3x &+ (D+2)y = -1 \end{aligned}$$

$3 \cdot (1) - (D+1)(2)$

$$3\cancel{(D+1)x} + 3(D-1)y = 6$$

$$-\cancel{3(D+1)x} - (D+1)(D+2)y = (D+1)1$$

$$[3\cancel{D} - 3 - (D^2 + \cancel{3D} + 2)]y = 6 + 0 + 1$$

$$(-D^2 - 5)y = 7 \Rightarrow (D^2+5)y = -7$$

$y'' + 5y = -7$ $\qquad y_c(t) = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t$

$y_p = A \Rightarrow A = -\frac{7}{5}$

$$y(t) = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t - \frac{7}{5}$$

$(2)\quad 3x + (D+2)y = -1 \Rightarrow x = -\frac{1}{3} - \frac{1}{3}(D+2)y$

$$-\frac{1}{3}Dy = \frac{\sqrt{5}}{3}c_1 \sin\sqrt{5}t - \frac{\sqrt{5}}{3}c_2 \cos\sqrt{5}t$$

$$-\frac{2}{3}y = -\frac{2}{3}c_1 \cos\sqrt{5}t - \frac{2}{3}c_2 \sin\sqrt{5}t + \frac{14}{15}$$

$$\boxed{\begin{aligned} x(t) &= \left(-\tfrac{2}{3}c_1 - \tfrac{\sqrt{5}}{3}c_2\right)\cos\sqrt{5}t + \left(\tfrac{\sqrt{5}}{3}c_1 - \tfrac{2}{3}c_2\right)\sin\sqrt{5}t + \tfrac{3}{5} \\ y(t) &= c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \tfrac{7}{5} \end{aligned}}$$

↑ for presentation

Alternative approach for finding x
(Assuming we already found y):

ple:

$$\begin{aligned} ① \quad (D+1)x &+ (D-1)y = 2 \\ ② \quad 3x &+ (D+2)y = -1 \end{aligned}$$

Now elim. y

$(D+2)① - (D-1)②$

$$(D+2)(D+1)x + \cancel{(D+2)(D-1)y} = (D+2)2 = 4$$

$$-\ 3(D-1)x - \cancel{(D-1)(D+2)y} = (D-1)1 = -1$$

$$(D^2 + \cancel{3D} + 2 - \cancel{3D} + 3)x = 3$$

$$(D^2+5)x = 3 \Rightarrow m = \pm\sqrt{5}i$$

$$x_c(t) = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t$$

$$x_p = A \Rightarrow A = \frac{3}{5}$$

$$x(t) = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t + \frac{3}{5}$$

$$y(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \frac{7}{5}$$

We have too many constants. To reconcile them: $x(t), y(t) \to ②$ ← arbitrary choice

ple:

$$\begin{aligned} ① \quad (D+1)x &+ (D-1)y = 2 \\ ② \quad 3x &+ (D+2)y = -1 \end{aligned}$$

$$3c_3 \cos\sqrt{5}t + 3c_4 \sin\sqrt{5}t + \frac{9}{5} - c_1\sqrt{5}\sin\sqrt{5}t$$
$$+ c_2\sqrt{5}\cos\sqrt{5}t + 2c_1\cos\sqrt{5}t + 2c_2\sin\sqrt{5}t - \frac{14}{5} = -1$$

cosine terms:

$3c_3 + c_2\sqrt{5} + 2c_1 = 0$

$c_3 = -\frac{2}{3}c_1 - \frac{\sqrt{5}}{3}c_2$

sine terms:

$3c_4 - c_1\sqrt{5} + 2c_2 = 0$

$c_4 = \frac{\sqrt{5}}{3}c_1 - \frac{2}{3}c_2$,

const:

$\frac{9}{5} - \frac{14}{5} = -1$ ✓

as before.

Example:
$$\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} &= e^t \\ -\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y &= 0 \end{aligned}$$

Example:

$$\begin{aligned} Dx & & +z &= e^t \\ (D-1)x & +Dy & +Dz &= 0 \\ x & +2y & +Dz &= e^t \end{aligned}$$

Example:

$$\begin{aligned}
\frac{dx}{dt} &= -x + z \\
\frac{dy}{dt} &= -y + z \\
\frac{dz}{dt} &= -x + y
\end{aligned}$$